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ON THE LIMITING FORM OF THE EQUATION OF ANISOTROPIC HEAT CONDUCTION IN A ROD

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The paper considers the problem of the asymptotically substantiated reduction of the three-dimensional, in coordinates, equation describing the process of heat propagation in an anisotropic material to a one-dimensional equation. As a heat-transfer region, a cylindrical rod of an arbitrary cross section was taken. It is assumed that the matrix of thermal diffusivity coefficients depends on the spatial coordinates. In the constructed equivalent heat-conduction equation, a certain effective heat-transfer coefficient is represented and formulas for its calculation have been obtained. Examples of the calculation have been considered.

Introduction. In practical applications of heat-and-mass transfer equations, situations are not infrequent where mathematical models of a process contain terms that considerably complicate the analysis of the problem by traditional methods of theoretical thermal physics, for example, the presence of convective transfer with a rather complex velocity field, the coordinate dependence of transfer coefficients, anisotropy of the medium's properties in terms of heat and mass propagation, etc. At the same time, researchers are often interested not in a detailed field of temperatures and concentrations, but, for example, in the mean value of temperature in the cross section of the channel, the value of the heat flow through the surface of a given boundary, and some of the other integral characteristics of the process. In such situations, an attractive point is the obtaining of simplified equations of the process that are fairly exact for practical needs. An example of such a simplification is the Taylor effective diffusion (heat conduction) model [1], which has found wide application in describing the heat and mass transfer in channels, devices, etc. For the cross-section mean concentration of substance, Taylor proposed an equation with an effective diffusion (dispersion) coefficient, which was calculated proceeding from the channel velocity profile. The principal point in the transformation of the problem proposed by Taylor is the fact that it can be substantiated by certain arguments (calculations) of the physical and mathematical kind. This turned out to be quite attractive for simplifying the mathematical description of the heat-and-mass exchange processes, and, therefore, the Taylor method was generalized and improved in various directions (e.g., [2–6]).

In the proposed problem (see below) (as in the Taylor problem [1]), direct averaging of initial equations does not lead to the desired result, since the terms making the averaged problem open remain in the equation. In such a situation, the main goal will be the development of averaging methods leading to a closed system of equations (an equation). Substantiated averaging by asymptotic methods also makes it possible to reveal the error of the fundamental relations and show further steps (if they are needed) to revise results. In a certain (ideological) sense, analogous problems were considered in [7, 8], where certain heat-conduction problems after asymptotic analysis acquired equations with a smaller number of independent variables than in the initial formulation. The establishment of the relationship between this calculation and the problem of minimizing a certain functional is useful for calculating the effective thermal diffusivity. Thus, in this work the possibility of using variation methods to determine the effective coefficient is shown.

In a three-dimensional space under proper conditions, there are two techniques of averaging the temperature field over either one or two coordinates. Earlier [7, 9], the heat transfer in a layer (in a stationary [7] and a nonstationary formulation [9]) was considered and an averaging transverse to it was performed. This led to an equivalent heat-conduction equation with two independent spatial coordinates. It should be noted that the properties of the heat transfer in the averaged model were described by the effective thermal diffusivity tensor. Here a more complicated problem connected with the asymptotic averaging over two coordinates will be analyzed, as a result of which we will

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arrive at a space-coordinate one-dimensional heat-conduction equation. In this case, the heat-transfer properties in the averaged model are determined by one parameter — effective thermal-diffusivity. Some examples of exact calculation of this coefficient will be considered and the possibility of generalizing the problem when the averaged heat-conduction equation becomes nonlinear will be shown.

Formulation of the Problem. The problem under consideration concerns the heat propagation in a rod made from a material that is anisotropic in terms of the heat transfer. As the edge of the rod, a closed surface of an arbitrary shape serves. For practical and mathematical purposes, it is enough to think of the curve in the cross section of the rod as consisting of a finite number of smooth curves. We will consider the problem in the domain of $Z \in (-\infty, +\infty)$, X, $Y \in \Omega$, where Ω is a domain in the plane (X, Y) bounded by the contour γ with the above requirement. The process of heat propagation in the rod is described by the equation

$$\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial X_\mu} \left(a_{\mu\nu} (X, Y, Z) \frac{\partial T}{\partial X_\nu} \right), \qquad (1)$$

where, as is often done in tensor and matrix calculi, summation from one to three with respect to the twice-repeated index (in this case μ and ν) is taken. Here for convenience $X_1 = X$, $X_2 = Y$, and $X_3 = Z$ are assumed; $a_{\mu\nu}(X, Y, Z)$ is the thermal diffusivity tensor, which we will assume to comply with the requirements of nonequilibrium thermodynamics [10, 11], namely, symmetry $a_{ij} = a_{ji}$ (Onsager relations) and be positive definite: $a_{ij}\xi_i\xi_j \geq \kappa\xi_i\xi_j$ $(\kappa > 0)$. The latter requirement is associated with the dissipativity of the process and entropy production [10, 11].

As boundary conditions of Eq. (1), we take

$$n_\mu a_{\mu\nu}\, \partial T / \partial X_\nu \big|_\gamma = 0\,, \qquad (2)$$

$$T \big|_{Z \to \pm\infty} < \infty\,. \qquad (3)$$

Here n_μ denotes cosines of angles formed by the vector of the unit normal **n** and the direction of the corresponding axis, with $n_z = n_3 = 0$. The boundary condition (2) expresses the absence of heat flow in the direction of the outer normal to the boundary and condition (3) — the restriction of the solution at a large distance from the origin of coordinates. The domain extent in the direction of the Z axis is of no fundamental importance here, as is the condition at the Z = const boundaries in the case of a finite domain. Condition (3) has been written for completeness of the formulation of the problem. It is also completed with the initial condition

$$T \big|_{\tau=0} = T_n (X, Y, Z)\,. \qquad (4)$$

Analysis of the Problem. The chief goal of the present paper is to simplify problem (1)–(4). The main additional condition is (2). Then certain complications in the formulation of problem (1)–(4) will be given, which also admit a simplified asymptotic formulation according to the scheme described.

Let us introduce the operation of averaging some quantity F:

$$\langle F (...) \rangle = \frac{1}{S} \int_S F (X, Y, ...)\, dS\,, \qquad (5)$$

where S is the area of the domain Ω and dots under the sign of the function F denote other variables (except X, Y) on which the function F may depend.

It should be noted that in the particular case of an isotropic medium (more precisely, a medium with zero nondiagonal tensor components $a_{\mu\nu} \neq 0$ at $\mu \neq \nu$ and with constant components $a_{\mu\mu}$), direct averaging of Eq. (1) and conditions (3), (4) with allowance for the boundary condition (2) leads to a one-dimensional heat-conduction problem for the cross-section mean temperature with the thermal diffusivity a_{zz}, and it is allowed to assume a_{zz} to be the function Z (one can also assume a_{xx} and a_{yy} to be the functions X, Y, and Z). We will be interested in a nontrivial more general case. However, we will use the specificity of the domain of integration of Eq. (1) where, with a domain that is noticeably more extended in the direction of the Z axis than in the directions of X and Y, the scales of the corresponding variables will differ in value. We will write Eq. (1) and the boundary condition (2) more explicitly according to the formulas

$$x_1 = x = \frac{X}{L_*}, \quad x_2 = y = \frac{Y}{L_*}, \quad x_3 = z = \frac{Z}{Z_*}, \quad A_{\mu\nu} = \frac{a_{\mu\nu}}{a_*}, \quad t = \tau \frac{a_*}{Z_*^2}, \quad \varepsilon = \frac{L_*}{Z_*}, \tag{6}$$

in the form

$$\varepsilon^2 \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial T}{\partial x_j}\right) + \varepsilon\left[\frac{\partial}{\partial x_i}\left(A_{iz}\frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial z}\left(A_{iz}\frac{\partial T}{\partial x_i}\right)\right] + \varepsilon^2 \frac{\partial}{\partial z}\left(A_{zz}\frac{\partial T}{\partial z}\right), \tag{7}$$

where hereinafter we will assume that summation from one to two, with respect to the twice-repeated Latin index, e.g., $A_{ij}\partial T/\partial x_j = \sum_{j=t}^{2} A_{ij}\partial T/\partial x_j$, is taken. If a Greek index (as in (1)) occurs twice, summation from one to three with respect to it is taken. We especially highlight the index on the Z coordinate; no summation can be made with respect to it.

The boundary condition (2) in variables (6) takes on the form

$$n_i A_{ij}\partial T/\partial x_j\big|_\gamma + \varepsilon n_i A_{iz}\partial T/\partial z\big|_\gamma = 0 . \tag{8}$$

The scale for the time in (6) was chosen in accordance with the fact that we will construct an effective heat-conduction equation with only one spatial coordinate z and, therefore, the formula for t is quite natural. Conditions (3) and (4) retain their form in the dimensionless variables (6), as does the formula for averaging (5). Therefore, we will not write them anew and will designate the dimensionless cross section area as s.

For the heat propagation in the rod, $\varepsilon << 1$, as a rule; therefore, a natural method for seeking a solution of problem (7), (8), (3), (4) will be the method of series expansion parameter perturbation [12, 13]. Thus, we seek a solution of the above problem in the form of the expansion

$$T = T_0\,(x, y, z, t) + \varepsilon T_1\,(x, y, z, t) + \varepsilon^2 T_2\,(x, y, z, t) + \ldots , \tag{9}$$

substituting which into Eq. (7) and the boundary condition (8), we obtain, after grouping the terms of the same order with respect to ε, the following sequence of problems:

$$\partial\,(A_{ij}\partial T_0/\partial x_j)/\partial x_i = 0\,, \quad n_i A_{ij}\partial T_0/\partial x_j\big|_\gamma = 0\,; \tag{10}$$

$$\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial T_1}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(A_{iz}\frac{\partial T_0}{\partial z}\right) = -\frac{\partial}{\partial z}\left(A_{iz}\frac{\partial T_0}{\partial x_i}\right), \quad n_i\left(A_{ij}\frac{\partial T_1}{\partial x_j} + A_{iz}\frac{\partial T_0}{\partial z}\right)\Bigg|_\gamma = 0\,;$$

$$\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial T_k}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(A_{iz}\frac{\partial T_{k-1}}{\partial z}\right) = \frac{\partial T_{k-2}}{\partial t} - \frac{\partial}{\partial z}\left(A_{zz}\frac{\partial T_{k-2}}{\partial z}\right) - \frac{\partial}{\partial z}\left(A_{iz}\frac{\partial T_{k-1}}{\partial x_i}\right); \tag{11}$$

$$n_i\left(A_{ij}\frac{\partial T_k}{\partial x_j} + A_{iz}\frac{\partial T_{k-1}}{\partial z}\right)\Bigg|_\gamma = 0\,, \quad k = 2, 3, \ldots . \tag{12}$$

Then, for the sake of brevity, we will designate the operator $\partial(A_{ij}\partial/\partial x_j)/\partial x_i$ as U. Multiply Eq. (10) by the function T_0 and integrate the result with respect to the cross section of the rod Ω. Using the Green theorem, we will obtain

$$0 = \langle T_0 U T_0 \rangle = \frac{1}{s} \oint_\gamma n_i T_0 A_{ij} \frac{\partial T_0}{\partial x_j} d\gamma - \left\langle A_{ij} \frac{\partial T_0}{\partial x_i} \frac{\partial T_0}{\partial x_j} \right\rangle . \tag{13}$$

By virtue of the boundary condition (10), the integral along the contour γ is equal to zero. Since the matrix A_{ij} is positive definite, by virtue of the assumed continuity of the components of the matrix A_{ij} and the quantities $\partial T_0/\partial x_j$ from Eq. (13), it follows that $\partial T_0/\partial x_j \equiv 0$, i.e., the variable T_0 is independent of coordinates x and y. Denote $T_0 = G(z, t)$. In view of this problem, (11) will be written as

$$UT_1 + (\partial A_{iz}/\partial x_i)\, \partial G/\partial z = 0 , \quad n_i\, (A_{ij} \partial T_1/\partial x_j + A_{iz} \partial G/\partial z) \big|_\gamma = 0 . \tag{14}$$

Let us give the solution of (14) in the form

$$T_1 = N\,(x, y)\, \partial G/\partial z + T_1^0\,(z, t) , \tag{15}$$

where the function N satisfies the relations

$$UN + \partial A_{iz}/\partial x_i = 0 , \quad n_i\, (A_{ij} \partial N/\partial x_j + A_{iz}) \big|_\gamma = 0 . \tag{16}$$

Then the function T_1^0, as can easily be seen, will satisfy problem (10), and the same calculations as for T_0 lead to the fact that the function T_1^0 will only depend on z and t, which we have just written in formula (15).

The problem of determining the function N is a Neumann problem, which, as is known [14], requires a special condition for its solvability, in this case one such as the following (obtained from condition (11) after elementary calculations):

$$\oint_\gamma n_i\, (A_{ij} \partial T_1/\partial x_j + A_{iz} \partial G/\partial z) \big|_\gamma\, d\gamma = 0 .$$

In our case, this requirement is satisfied by virtue of the boundary condition (14). Despite this fact, the solution of problem (16) is not unique (the addition of a constant to N does not change the formulation of the problem [14]). To obtain a unique function N, it is enough to impose on N an additional requirement, $\langle N \rangle = 0$. In such a case, the physical meaning of the term $T_1^0(z, t)$ in formula (15) will be its equality to the mean value of the function T_1 of the first approximation in the number ε: $T_1^0(z, t) = \langle T_1 \rangle$.

Each equation (11) and (12) for the T_k functions has a certain necessary condition for a solution to exist. Denote the sum of the terms on the right-hand side of Eq. (12) by the symbol F_k. Then, averaging the equation $UT_k + \partial(A_{iz} \partial T_{k-1}/\partial z)/\partial x_i = F_k$, using the Green theorem, and taking into account the boundary condition (12), we obtain

$$\langle F_k \rangle = 0 , \quad k = 2, 3, \ldots . \tag{17}$$

Here we will consider only the equation for the basic approximation of expansion (9), for which it suffices to take in (17) $k = 2$, taking into account the obtained dependences for variables T_0 and T_1.

From relations (17) and (15) we have

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial z} \left(\langle A_{zz} \rangle \frac{\partial G}{\partial z} \right) + \frac{\partial}{\partial z} \left(\left\langle A_{iz} \frac{\partial N}{\partial x_i} \right\rangle \frac{\partial G}{\partial z} \right), \tag{18}$$

where it is taken into account that T_1^0 depends on x_i ($i = 1, 2$). From the form of Eq. (18) the formula for the effective thermal diffusivity follows:

$$A_0\,(z) = \langle A_{zz} + A_{iz} \partial N/\partial x_i \rangle . \tag{19}$$

Derivation of Different Formulas for Effective Thermal Diffusivity. Proceeding from the physical meaning, it is essential that the effective thermal diffusivity $A_0(z)$ be positive. This is not apparent from formula (19). Let us transform it as follows:

$$A_0(z) = \langle A_{zz} \rangle + \langle A_{iz} \partial N / \partial x_i - N (UN + \partial A_{iz} / \partial x_i) \rangle = \langle A_{zz} \rangle + 2 \langle A_{iz} \partial N / \partial x_i \rangle - \langle NUN \rangle -$$

$$- \langle \partial (A_{iz} N) / \partial x_i \rangle = \langle A_{zz} \rangle + 2 \langle A_{iz} \partial N / \partial x_i \rangle + \langle A_{ij} (\partial N / \partial x_i)(\partial N / \partial x_j) \rangle -$$

$$- \frac{1}{s} \oint_{\gamma} n_i N (A_{ij} \partial N / \partial x_j + A_{iz}) \big|_{\gamma} d\gamma , \qquad (20)$$

where we added to formula (19), according to Eq. (16), a nonzero term and made use of the Green theorem similarly to the calculations in (13) for the term $\langle NUN \rangle$ and directly for $\langle \partial (A_{iz} N) / \partial x_i \rangle$. By virtue of the boundary condition (16), the integral along the line γ in expression (20) is equal to zero and, therefore, we finally have

$$A_0(z) = \langle A_{zz} + 2 A_{iz} \partial N / \partial x_i + A_{ij} (\partial N / \partial x_i)(\partial N / \partial x_j) \rangle . \qquad (21)$$

From formula (21) it can easily be seen that $A_0(z) > 0$. Indeed, consider the vector ξ with components $\xi_1 = \partial N / \partial x_1$, $\xi_2 = \partial N / \partial x_2$, $\xi_3 = 1$. Then, taking into account the symmetry of coefficients $A_{\mu\nu}$: $A_{\mu\nu} = A_{\nu\mu}$, we notice that under the averaging sign in expression (21) there is a positive-definite quadratic form $A_{\mu\nu} \xi_\mu \xi_\nu$, which in the case under consideration is strictly positive, since the vector ξ contains a component $\xi_3 = 1$ known to be nonzero. The integration (averaging) in dependence (21) of the strictly positive continuous function preserves positivity; therefore $A_0(z) > 0$. Moreover, we obtain the following estimates for the coefficient $A_0(z)$:

$$A_0(z) \geq \kappa \langle (\partial N / \partial x)^2 + (\partial N / \partial y)^2 + 1 \rangle \geq \kappa , \qquad (22)$$

where κ is a constant in the relation of the positive definiteness of the matrix $A_{\mu\nu}$: $A_{\mu\nu} \xi_\mu \xi_\nu \geq \kappa \xi_\mu \xi_\mu$.

In deriving expression (21) from relation (20), we can see (which also follows from comparison of formulas (19) and (21)) that the equality

$$\langle A_{iz} \partial N / \partial x_i \rangle = - \langle A_{ij} (\partial N / \partial x_i)(\partial N / \partial x_j) \rangle , \qquad (23)$$

holds. Its substitution into (19) yields another useful expression for the effective thermal diffusivity:

$$A_0(z) = \langle A_{zz} - A_{ij} (\partial N / \partial x_i)(\partial N / \partial x_j) \rangle . \qquad (24)$$

Formula (21) is also useful due to the fact that it is closely allied to the possibility of using variational calculation methods to find the function $A_0(z)$. Indeed, treating (21) as a functional depending on the unknown function $N(x, y)$ and calculating the first variation, we obtain, as an Euler equation [14–16], Eq. (16) with the corresponding boundary condition, which is usually referred to as the natural boundary condition. Thus, taking into account the positive definiteness of functional (21), we find that the value of the coefficient $A_0(z)$ is the least possible value for the quadratic functional (21). In this connection, note that usually [15] variational methods yield better results in determining integral characteristics (as in our case) than for the sought function distribution over the domain of functional determination. Consequently, minimization of functional (21) by conventional methods [15, 16] (e.g., by the Ritz method) seems to be a promising method for calculating the effective thermal diffusivity $A_0(z)$.

We give another formula for calculating the coefficient $A_0(z)$, which in a number of cases permits obtaining exact analytical results. Consider the Sturm–Liouville spectral problem

$$UP_k + \lambda_k P_k = 0 , \quad n_i A_{ij} \partial P_k / \partial x_j \big|_{\gamma} = 0 , \quad k = 0, 1, \dots , \qquad (25)$$

where λ_k at $k = 0, 1, 2, \dots$ denotes the eigenvalues of problem (25), with the least eigenvalue being $\lambda_0 = 0$; in this case, $P_0 = \text{const} = 1$ and the other $\lambda_k > 0$ and in the general case may depend on the z coordinate as on the parameter. The functions $P_k(x, y, z)$, according to the general theory [14, 17], will be orthogonal (they can be normalized): $\langle P_i P_j \rangle = \delta_{ij}$, where $\delta_{ij} = 1$ at $i = j$ and $\delta_{ij} = 0$ at $i \neq j$ is the Kronecker symbol. By virtue of the fact that $\lambda_0 = 0$ is an eigenvalue, we have $\langle P_j \rangle = 0$ at $j \neq 0$ (the latter also follows from the averaging of Eq. (25) at $\lambda_k \neq 0$).

Let us express the solution of problem (16) as a system of functions P_k. In so doing, we will not use the function P_0 for the uniqueness condition of the solution $\langle N \rangle = 0$ to be met. Multiply Eq. (16) by the function P_k $(k > 0)$, average the result, and "integrate by parts" (using the Green function) twice. We obtain a chain of inequalities:

$$\langle P_k \partial A_{jz} / \partial x_j \rangle = - \langle P_k U N \rangle = - \frac{1}{s} \oint\limits_{\gamma} d\gamma n_i P_k A_{ij} \partial N / \partial x_j + \langle A_{ij} (\partial P_k / \partial x_i) (\partial N / \partial x_j) \rangle =$$

$$= \frac{1}{s} \oint\limits_{\gamma} n_i A_{ij} (N \partial P_k / \partial x_j - P_k \partial N / \partial x_j) \, d\gamma - \langle N U P_k \rangle ,$$

from which, applying the Green function to the left-hand side and using relations (16) and (25), we finally obtain

$$\langle N P_k \rangle = - \alpha_k / \lambda_k , \quad \alpha_k = \langle A_{jz} \partial P_k / \partial x_j \rangle . \tag{26}$$

Now, substituting the formula expansion of the function

$$N = \sum_{k=1}^{\infty} \langle N P_k \rangle \, P_k = - \sum_{k=1}^{\infty} \alpha_k P_k / \lambda_k$$

into expression (19), we obtain the sought dependence

$$A_0 (z) = \langle A_{zz} \rangle - \sum_{k=1}^{\infty} \alpha_k^2 / \lambda_k . \tag{27}$$

Since $A_0(z) > 0$, the sum in (27) is limited by the number $\langle A_{zz} \rangle$ ($A_{zz} > 0$ by virtue of the Sylvester criterion [18] of the positive definiteness of the quadratic form $A_{\mu\nu} \xi_\mu \xi_\nu$).

The formula for the coefficient α_k contains the j sum. In a particular case, if the values of A_{jz} are expressed as a gradient in the A_{ij} metric, i.e., $A_{jz} = A_{ij} \partial \varphi / \partial x_i$, then the use of the Green theory and conditions (25) leads to inequalities $\alpha_k = \langle A_{jz} \partial P_k / \partial x_j \rangle = -\langle \varphi U P_k \rangle = \lambda_k \langle \varphi P_k \rangle$. In so doing, a formula for α_k containing no summation has been obtained. As a result, in relation (27) instead of α_k^2 / λ_k we will have $\lambda_k (\varphi P_k)^2$ under the summation sign. Naturally, the introduced function φ should not disturb the positive definiteness of the $A_{\mu\nu}$ matrix. This imposes certain conditions, but we will not give them.

Note that instead of (25) we could consider another problem; the most natural of the possible ones is closely associated with the replacement of the boundary condition (25) by a condition of the third kind: $n_i A_{ij} \partial P_k / \partial x_j \big|_\gamma = -n_i A_{iz} P_k \big|_\gamma$. However, variant (25) seems to be better, since it does not include the A_{iz} function in the boundary condition and the final formula (27) is much simpler.

The asymptotic analysis performed shows that the rod made from an anisotropic material behaves as an ordinary homogeneous body in studying heat conduction in only one direction.

Determination of the Initial Condition for Eq. (18). To completely formulate the problem, it is necessary to formulate the initial condition for Eq. (18), which we write in the final form

$$\frac{\partial G}{\partial t} = \frac{\partial}{\partial z} \left(A_0 (z) \frac{\partial G}{\partial z} \right). \tag{28}$$

In so doing, note that actually we have constructed an "external" [12, 13] expansion suitable for describing the process at fairly large values of time. The special (singular) behavior of this expansion is evidenced by the absence of boundary condition (4) from the formulation of the problems for T_j functions. It dropped out because of the large ($\sim 1/\varepsilon^2$) scale chosen for time. To describe the behavior of the solution at small values of time, it is necessary to introduce a "compressed" time $\zeta = t/\varepsilon^2$ and construct a new expansion [12, 13]. The new "outer" [12, 13] problem is written as follows:

$$\frac{\partial T}{\partial \zeta} = UT + \varepsilon \left[\frac{\partial}{\partial x_i} \left(A_{iz} \frac{\partial T}{\partial z} \right) + \frac{\partial}{\partial z} \left(A_{iz} \frac{\partial T}{\partial x_i} \right) \right] + \varepsilon^2 \frac{\partial}{\partial z} \left(A_{zz} \frac{\partial T}{\partial z} \right), \tag{29}$$

$$n_i A_{ij} \partial T/\partial x_j \big|_\gamma + \varepsilon n_i A_{iz} \partial T/\partial z \big|_\gamma = 0 \,, \quad T \big|_{\zeta=0} = T_n (x, y, z) \,.$$

As before, we restrict ourselves to the principal approximation of the inner expansion

$$T = \overline{T}_0 (x, y, z, \zeta) + \varepsilon \overline{T}_1 (x, y, z, \zeta) + ... \,, \tag{30}$$

where the bar over T denotes the inner solution. The equation for the $\overline{T}_0$ function of the principal approximation is obtained by substituting into (29) the value of $\varepsilon = 0$. This leads to the problem

$$\partial \overline{T}_0/\partial \zeta = U \overline{T}_0 \,, \quad n_i A_{ij} \partial \overline{T}_0/\partial x_j \big|_\gamma = 0 \,, \quad \overline{T}_0 \big|_{\zeta=0} = T_n (x, y, z) \,. \tag{31}$$

For our purposes (joining with the solution of the outer problem (9)) it will be enough to determine only the mean value $\langle \overline{T}_0 \rangle$ of the function $\overline{T}_0$. To this end, we average Eq. (31). As before, we obtain the mean value of the operator on the right-hand side equal to zero, from which we find

$$\langle \partial \overline{T}_0/\partial \zeta \rangle = 0 \to \langle \overline{T}_0 \rangle = \text{const} (\zeta, x, y) = \langle T_n \rangle \,, \tag{32}$$

where we made use of the initial condition (31). Now, using (32), we apply the principle of limiting joining [12, 13] for the means of the principal terms of the outer and inner expansions:

$$\lim_{t \to 0} \langle T_0 \rangle = \lim_{t \to 0} G (z, t) = G \big|_{t=0} = \lim_{\zeta \to \infty} \langle \overline{T}_0 \rangle = \langle T_n \rangle = G_n (z) \,. \tag{33}$$

Thus, the function $T_0(z, t)$ at the initial instant of time is equal to the mean value of the assumed function of the initial problem. This would be expected intuitively.

The inner problem describes the fast ($t = O(\varepsilon^2)$) process of temperature equalization in the cross section of the rod. This means that the dimensionless homogenization time in the system is of the order of the number of ε^2.

The chief problem in formulating the effective heat-conduction equation (18) or (28) is the calculation of the coefficient A_0 by one of the formulas (19), (21), (24), (27), since the methods for integrating Eq. (28) have been well developed. In turn, this problem is reduced to the determination of the function N. It is clear that in the case of a rather complex, as to the shape, region of the rod's cross section and of the coordinate dependence of the matrix function A_{ij}, it is hardly probable to find an analytical solution of the problem. As mentioned above, variational and projection methods [15, 16], which usually determine the integral characteristics of the solution more exactly than the solution itself, seem to be promising here. Consider a few examples, where A_0 is calculated analytically and reduces to simple quadratures.

Example 1. Consider a flat layer $x \in (0, a)$, $y \in (-\infty, +\infty)$, $z \in (-\infty, +\infty)$ in the absence of the dependence of the components of the matrix $A_{\mu\nu}$ on the variable y; more precisely, assume that $A_{xy} = A_{yz} = 0$ and the other components of $A_{\mu\nu}$ depend only on x and z. In such a case, the problem of determining the function N has the form

$$\frac{d}{dx} A_{xx} \frac{dN}{dx} + \frac{dA_{xz}}{dx} = 0 \,, \quad A_{xx} \frac{dN}{dx} + A_{xz} \bigg|_{x=0;a} = 0 \,. \tag{34}$$

In relations (34), we write the total derivatives with respect to x, since the dependence on z is parametric. The first integral of Eq. (34) gives for the function N the expression $dN/dx = -A_{xz}/A_{xx}$. Substituting it into formula (19), we arrive at the dependence for the effective thermal diffusivity:

$$A_0(z) = \langle A_{zz} \rangle - \langle A_{xz}^2 / A_{xx} \rangle = \frac{1}{a} \int_0^a \frac{(A_{xx} A_{zz} - A_{xz}^2)\, dx}{A_{xx}} . \tag{35}$$

In the numerator of the subintegral expression (35), there is a minor of the matrix **A**, which, as the denominator A_{xx}, is positive by virtue of the positive definiteness **A** (Sylvester criterion [18]).

The ease of analysis of Example 1 is due to the fact that the problem of determining the function N is one-dimensional. Consider a case more general than problem (34) which admits, nevertheless, reduction to a problem analogous to (34).

Example 2. Let the rectangle $x \in (0, a)$, $y \in (0, b)$ be the domain of the rod cross section and the following conditions (restrictions) be imposed on the coefficients of the thermal diffusivity matrix: elements A_{xx}, A_{xy}, A_{xz}, and A_{yz} depend only on x and z, and between these elements the equality $\Delta_{yz} = A_{xx} A_{yz} - A_{xy} A_{zz}$ is realized. It is clear that from these conditions, as a particular case, the conditions of the first example can be obtained. Denote the only y-averaged value by a "cap" above, i.e., $\hat{F} = \frac{1}{b} \int_0^b F(x, y, z, t) dy$. Applying the above averaging to the equation for the function N and the boundary condition on the variable x

$$A_{xx} \partial N / \partial x + A_{xy} \partial N / \partial y \big|_{x=0;a} = - A_{xz} , \tag{36}$$

we arrive at the expressions

$$\frac{d}{dx} A_{xx} \frac{d\hat{N}}{dx} + N \frac{dA_{xy}}{dx} \Bigg|_{y=0}^{y=b} + \frac{dA_{xz}}{dx} = 0 , \tag{37}$$

$$\left(A_{xx} \frac{d\hat{N}}{dx} + N A_{xy} \Bigg|_{y=0}^{y=b} + A_{xz} \right) \Bigg|_{x=0;a} = 0 , \tag{38}$$

where the boundary condition for the function N at the boundaries $y = b$ and $y = 0$

$$A_{xy} \partial N / \partial x + A_{yy} \partial N / \partial y \big|_{y=0;b} = - A_{yz}$$

has been used. Note that because we use, as the boundary of the domain Ω, the rectangle formed by the segments parallel to the coordinates of the axes; the coordinates of the normal vectors are simply written on the rectilinear segments, and we have the above boundary conditions following from (16).

The first integral of Eq. (37) satisfying the conditions of (38) will be the following one:

$$A_{xx} \frac{d\hat{N}}{dx} + N A_{xy} \Bigg|_{y=0}^{y=b} + A_{xz} = 0 . \tag{39}$$

We now transform expression (19) for the quantity $A_0(z)$ in view of the statements assumed for the coefficients $A_{\mu\nu}$ and Eq. (39). We have

$$A_0(z) = \langle A_{zz} + A_{xz} \partial N / \partial x + A_{yz} \partial N / \partial y \rangle = \langle A_{zz} \rangle + \frac{1}{a} \int_0^a dx \left\{ A_{xz} d\hat{N} / dx + A_{yz} N \big|_{y=0}^{y=b} \right\} =$$

$$= \langle A_{zz} \rangle - \frac{1}{a} \int_0^a dx A_{xz}^2 / A_{xx} - \frac{1}{a} \int_0^a dx N \Delta_{yz} / A_{xx} \big|_{y=0}^{y=b} = \langle (A_{xx} A_{zz} - A_{xz}^2) / A_{xx} \rangle , \tag{40}$$

since $\Delta_{yz} = A_{xx}A_{yz} - A_{xy}A_{zz} = 0$. Using this equality in the form $A_{zx}/A_{xx} = A_{yz}/A_{xy}$, we can put formula (40) into a form symmetric about the indices x and y:

$$A_0(z) = \langle (A_{xy}A_{zz} - A_{yz}A_{xz})/A_{xy} \rangle \; . \tag{41}$$

In so doing, it is assumed that the element A_{xy} is not equal to zero (A_{xx}, as a diagonal element of the positive-definite matrix, is always positive). Note that although we have obtained a formula symmetric about the indices x and y, the conditions of Example 2 are nonsymmetric. We can easily obtain a formula instead of (40) by interchanging the role of x and y ($x \leftrightarrow y$) in the formulation of the problem. Formula (41) will thereby retain its form.

Example 3. As an example of using formula (27) to calculate $A_0(z)$, consider the same domain as in Example 2, but assume that the coefficients $A_{\mu\nu}$ satisfy the following conditions: $A_{xx} = A_{yy} = A(z)$, $A_{xy} = 0$. The other $A_{\mu\nu}$ components can be arbitrary.

Since the variables in the equation for the function N (if it is considered to be homogeneous, i.e., $UN = 0$)

$$A(z)\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right) = -\frac{\partial A_{xz}}{\partial x} - \frac{\partial A_{yz}}{\partial y}\,, \qquad A\left.\frac{\partial N}{\partial x}\right|_{x=0;a} = -A_{xz}\,, \quad A\left.\frac{\partial N}{\partial y}\right|_{y=0;b} = -A_{yz}\,, \tag{42}$$

are separated, the eigenfunctions in terms of which the solution is expressed in the form of (27) (of the type $P_k(x, y, z)$ in (25)) will be given as a product of functions separately depending on the variables x and y and the single series will be replaced by a double one [16, 17]. In this example, we will use dimensional variables and take, as a set of eigenfunctions, the following ones:

$$R_{kn}(x, y) = 4\cos(\pi x k/a)\cos(\pi y n/b)\,, \quad k, n > 1 \tag{43}$$

(we will not need functions with zero values of indices k and n). They satisfy the normalization condition

$$\langle R_{kn}R_{lm} \rangle = \delta_{kl}\delta_{mn}\,. \tag{44}$$

Multiplying Eq. (42) by the function R_{mn} and averaging the result, after some calculations using the Green theory, the boundary conditions (42), and the equality to zero of the partial derivatives of the function R_{mn} at the domain boundary, we obtain the formula for the Fourier coefficient of the function N:

$$\langle NR_{mn} \rangle = \rho_{mn} = \left\langle A_{xz}\frac{\partial R_{mn}}{\partial x} + A_{yz}\frac{\partial R_{mn}}{\partial y} \right\rangle \Big/ \lambda_{mn}\,, \quad \lambda_{mn} = \pi^2 A\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right). \tag{45}$$

Now, substituting the Fourier series for the function

$$N = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \rho_{mn}R_{mn}/\lambda_{mn}$$

into formula (19), we find the sought expression for the effective thermal diffusivity:

$$A_0(z) = \langle A_{zz} \rangle - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \rho_{mn}^2/\lambda_{mn}\,. \tag{46}$$

The system of functions (43) can also be used to solve a problem more general than (42), when $A_{xx} \neq A_{yy}$, but both these values depend only on z. It can easily be seen that in this case formula (46) can be replaced by the expression

$$A_0(z) = \langle A_{zz} \rangle - \frac{1}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \rho_{mn}^2 \Big/ \left[A_{xx}(z)\,(m/a)^2 + A_{yy}(z)\,(n/b)^2 \right]. \qquad (47)$$

Conclusions. The proposed derivation of the effective thermal diffusivity is naturally generalized to the presence of heat sources in the bulk of the rod and the inflow of heat through its lateral surface. If the scales of values of these sources are consistent with the characteristic time scale of the problem, more exactly, in the dimensionless variables (6), they have the order of ε^2, i.e., the term $-\varepsilon^2 Q(T, x, y, z)$ will enter into the right-hand side of Eq. (7) and the boundary condition (8) will take the form $n_i A_{ij} \partial T/\partial x_j \big|_\gamma + \varepsilon n_i A_{iz} \partial T/\partial z \big|_\gamma = -\varepsilon^2 q(T, \gamma, z)$; then the effective thermal diffusivity equation can be replaced by the expression

$$\frac{\partial G}{\partial t} + \langle Q(G, z) \rangle + \frac{1}{s} \oint_\gamma d\gamma q(G, \gamma, z) = \frac{\partial}{\partial z} \left(A_0(z) \frac{\partial G}{\partial z} \right) \qquad (48)$$

with the same value of the effective thermal diffusivity, which was found previously. It is clear that the averaged heat source Q will depend on the function G and the coordinate z. And these arguments are left in the heat-source function in relation (48), which is essentially a nonlinear equation of a rather more general form for the one-dimensional case.

Note certain simplifications in solving the asymptotic problem (28), (3), (33) compared to the complete formulation (7), (8), (3), (4).

1. The presence of the series expansion parameter ε^2 at the time derivative in (7) and z derivatives inhibits the numerical search for a solution. The scales of the variables t in (28) are much larger than in x and y. They are adequate for the problem under consideration.

2. The asymptotic expansion makes it possible to split the solution of the initial problem into two: the first problem is to determine the effective thermal diffusivity by formulas (16) and (19) and the second problem is to determine the temperature field by the one-dimensional equation (28). Note that the non-one-dimensionality of the temperature field will show up subsequent approximations (in particular, formula (15) after the function T_1^0 is found). In the initial problem, in solving numerically, we will just the same have to use the operator U (its finite-difference approximation). In so doing, the "separation" of variables x and y on the one hand and z and t on the other is much more problematic.

3. The coefficient A_0 can be determined by variational calculation methods.

NOTATION

a and b, dimensions of the heat-transfer region in the examples; $A_{\mu\nu}$ and $\alpha_{\mu\nu}$, dimensionless and dimensional thermal diffusivity tensors, respectively ($\mu, \nu = 1, 2, 3$); a_*, scale of the thermal diffusivity tensor; A_0, effective thermal diffusivity; G, first term of the ε expansion of temperature; L_*, scale for variables X, Y; N, auxiliary function (16); S and s, dimensional and dimensionless areas of the rod cross section; P_k, eigenfunctions of problem (25); T, temperature; T_j, terms of ε expansion of temperature (9); $\overline{T}_j$, components of inner ε expansion of temperature (30); t, dimensionless time; $U = \partial(A_{ij}\partial/\partial x_j)/\partial x_i$, differential operator; X_μ and x_μ, dimensional and dimensionless Cartesian coordinates ($\mu = 1, 2, 3$); Z_*, scale for the variable Z; γ, line limiting the rod cross section; $\varepsilon = L_*/Z_*$, series expansion parameter; ξ, inner time; τ, dimensional time; $\langle\,\rangle$, averaging sign.

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